# Non Commutative Fourier Transform and Partial Differential Equation on the Motion Group

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May 23, 2014

**Abstract**

Let $V$ be the $n-$ dimensional real vectoriel group, $K$ a connected compact Lie group and $G = V \rtimes_\rho K$ be the motion group, which is the semi-direct product of the group $V$ and $K$. Let $\mathcal{U}$ be the enveloping algebra of $G$, which is the algebra of all the invariant partial differential equations on $G$. In this paper, we will define the Fourier transform on $G$ and we demonsrate the Plancherel theorem. Besides we give necessary and sufficient condition for the existence of a fundemantal solution for any invariant partial differential equations $P$ on $G$.

**Keywords**: **Key words :** Motion Group, Semidirect Product of Two Lie Groups, Fourier Transform, Plancherel Theorem, Invariant Partial Differential Operators.

**AMS 2000 Subject Classification:** $43A30\&35D\ 05$

# 1 Introduction and Results.

**1.** Let $V$ be the $n-$ dimensional real vectoriel group, $K$ a compact Lie group and $\rho : K \rightarrow GL(V)$ a continuous linear representation from $K$ in $V$ . Let $G = V \rtimes_\rho K$ be the motion group, which is the semi-direct product of the group $V$ and $K$. We supply $V$ by $K-$invariant scalar product which is denoted by $\langle \ , \rangle$. Let $\mathcal{S}(V)$ be the Schwartz space of $V$ . We denote by $\mathcal{S}(G)$ the complemented of the space $\mathcal{S}(V) \otimes C^\infty(K)$ tensor product of $\mathcal{S}(V)$ and $C^\infty(K)$. The topology of the space $\mathcal{S}(G)$ which is defined by the family of semi-normas

$$\partial^l_{\alpha,\beta}(f) = \sup_{|\alpha \leq p|,} \sup_{(v,y)\in V\times K} (1+|v|^2)^\beta \left\| Q^\alpha_v D^l f(v,y) \right\|_2 \tag{1}$$

turns $\mathcal{S}(G)$ a Frechet space wich can be called the Schwartz space of $G$, where $|\ |$ signifies the norm associated to (ı), see [3].

**Definition 1.1.** Let $P$ be an invariant differential operator on a connected Lie group $H$. by definition $P$ is said to be semi-globally solvable if there exist a distribution $T$ on $H$ such that $PT = \delta_H$, where $\delta_H$ ias the Dirac measure at the identity element of $H$.

**Definition 1.2.** Let $P$ be an invariant differential operator on a connected Lie group $H$ by definition $P$ is said to be globally solvable if for any function $g \in C^\infty(H)$, there exist a function $f \in C^\infty(H)$ such that

$$Pf = g \tag{2}$$

For all the following notations and results, see [2]

**2.** Let $\underline{k}$ be the Lie algebra of $K$ and $(X_1, X_2, \ldots.., X_m)$ a basis of $\underline{k}$ , such that the both operators

$$\Delta = \sum_{i=1}^{m} X_i^2 \tag{3}$$

$$D_q = \sum_{0\leq l\leq q} \left( -\sum_{i=1}^{m} X_i^2 \right)^l \tag{4}$$

are left and right invariant (bi-invariant) on $K$, this basis exist see $[2, p.564)$. For $l \in \mathbb{N}$, let $D^l = (1-\Delta)^l$, then the family of semi-norms $\{\gamma_l,\ l \in \mathbb{N}\}$ such that

$$\gamma_l(f) = (\int_K \left| D^l f(y) \right|^2 dy)^{\frac{1}{2}}, \qquad f \in C^\infty(K) \tag{5}$$

define on $C^\infty(K)$ the same topology of the Frechet topology defined by the semi-norms $\|X^\alpha f\|_2$ defined as

$$\|X^\alpha f\|_2 = (\int_K |X^\alpha f(y)|^2 \, dy)^{\frac{1}{2}}, \qquad f \in C^\infty(K) \tag{6}$$

where $\alpha = (\alpha_1, \ldots\ldots, \alpha_m) \in \mathbb{N}^m$, see $[2, p.565]$

**3.** Let $\widehat{K}$ be the set of all irreducible unitary representations of $K$. If $\gamma \in \widehat{K}$, we denote by $E_\gamma$ the space of representaion $\gamma$ and $d_\gamma$ its demension then we get

$$1 \leq \langle D_q tr\gamma(r), tr(r)\rangle = d_q(\gamma) \tag{7}$$

If $u(\eta)$ is a polynomial on $\mathbb{R}^n$ valued in $E_\gamma$, we denote by $\widetilde{u}(\eta)$ the matrix defined by

$$[\widetilde{u}_{ij}(\eta)] \tag{8}$$

where

$$\widetilde{u}_{ij}(\eta) = \left(\sum_{|\alpha| \leq m} \left|u_{ij}(\eta)^{(\alpha)}\right|^2\right)^{\frac{1}{2}} \tag{9}$$

$\alpha = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_m)$, $1 \leq \alpha_j \leq n$, $\eta^\alpha = \eta_{\alpha_1} \ldots\ldots \eta_{\alpha_m}$, and $(\eta_1, \eta_2, \ldots., \eta_n)$ is basis of $\mathbb{R}^n$. In this case we get.

$$\|\widetilde{u}(\eta)\|^2 = \sum_{|\alpha| \leq m} \left\|u(\eta)^{(\alpha)}\right\|^2 \tag{10}$$

**Definition 1.3.** *The Fourier transform of a function $f \in C^\infty(K)$ is defined as*

$$Tf(\gamma) = \int_K f(x)\gamma(x^{-1})dx \tag{11}$$

*where $T$ denotes the Fourier transform on $K$*

**Theorem (A. Cerezo) 1.1.** *Let $f \in C^\infty(K)$, then we have the inversion of the Fourier transform*

$$f(x) = \sum_{\gamma \in \widehat{K}} d\gamma tr[Tf(\gamma)\gamma(x)] \tag{12}$$

$$f(I) = \sum_{\gamma \in SO(3)} d\gamma tr[Tf(\gamma)] \tag{13}$$

*and the Plancherel formula*

$$\|f(x)\|_2^2 = \int |f(x)|^2 \, dx = \sum_{\gamma \in \widehat{K}} d_\gamma \, \|Tf(\gamma)\|_{H.S}^2 \tag{14}$$

*where $I$ is the identity element of $K$,where* $\|Tf(\gamma)\|_{H.S}^2$ is the norm of the Hilbert-Schmidt of the operator $Tf(\gamma)$

# 2 Fourier Transform on the Motion Group

**Definition 2.1.** *For every function $f$ belongs to $L^1(V \times K)$, one can define the Fourier tran*sform of $f$ by the following manner

$$T\mathcal{F}f(\xi,\gamma) = \int_V \int_K f(v,x) e^{-\, i\langle\, \xi,\, v\rangle} \, \gamma(x^{-1}) dv dx \tag{15}$$

*for all $\xi \in V \simeq V^*$ and for all $\gamma \in \widehat{K}$, where $\mathcal{F}$ is the partial Fourier transform on the real vector Lie group$V$*

Then we get the the Fourier inversion

$$f(v,x) = \sum_{\gamma \in \widehat{K}} d_\gamma \int_V tr[T\mathcal{F}f(\xi,\gamma) e^{\, i\langle\, \xi,\, v\rangle} \, \gamma(x^{-1})] d\xi \tag{16}$$

$$= \int_V \sum_{\gamma \in \widehat{K}} d_\gamma tr[T\mathcal{F}f(\xi,\gamma) e^{\, i\langle\, \xi,\, v\rangle} \, \gamma(x^{-1})] d\xi \tag{17}$$

$$f(0,1) = \sum_{\gamma \in \widehat{K}} d_\gamma \int_V tr[T\mathcal{F}f(\xi,\gamma)] d\xi \tag{18}$$

When $K = SO(n)$,our result is

**Theorem 2.1.** ***(Plancheral's formula)*** *For any $f \in L^1(G) \cap L^2(G)$,we get*

$$f \, * \overset{\vee}{f}(0,1) = \int_G |f(v,x)|^2 \, dv dx = \sum_{\gamma \in \widehat{K}} d_\gamma \int_V \|T\mathcal{F}f(\xi,\gamma)\|_2^2 \, d\xi \tag{19}$$

*where* $\overset{\vee}{f}$ *is the function defined by*

$$\overset{\vee}{f}(v,x) = \overline{f(v,x)^{-1}} \tag{20}$$

*Proof:*

$$\begin{aligned}
&\overset{\vee}{f} * f(0,1) \\
= &\sum_{\gamma\in\widehat{K}} d_\gamma tr \int_V T\mathcal{F}(\overset{\vee}{f} * f\ )(\xi,\gamma) d\xi \\
= &\sum_{\gamma\in\widehat{K}} d_\gamma \int_V \int_V \int_K tr[\overset{\vee}{f} * f(v,x))e^{-\ i\langle\ \xi,\ v\rangle}\ \gamma(x^{-1})] dvdxd\xi \\
= &\sum_{\gamma\in\widehat{K}} d_\gamma \int_V \int_V \int_K tr[\int_V \int_K f((u,y)^{-1}(v,x))\overset{\vee}{f}(u,y)e^{-\ i\langle\ \xi,\ v\rangle}\ \gamma(x^{-1})] dvdxdudyd\xi \\
= &\sum_{\gamma\in\widehat{K}} d_\gamma \int_V \int_V \int_K tr[\int_V \int_K f((u,y)(v,x))\overline{f(u,y)^{-1}}e^{-\ i\langle\ \xi,\ v\rangle}\ \gamma(x^{-1})] dvdxdudyd\xi \\
= &\sum_{\gamma\in\widehat{K}} d_\gamma \int_V \int_V \int_K tr[\int_V \int_K f(u+yv,yx)\overline{f(u,y)}e^{-\ i\langle\ \xi,\ v\rangle}\ \gamma(x^{-1})] dvdxdudyd\xi
\end{aligned}$$

Chinging variables $u + yv = w$ , $yx = z$, we have

$$\begin{aligned}
&\overset{\vee}{f} * f(0,1) \\
= &\sum_{\gamma\in\widehat{K}} d_\gamma \int_V \int_V \int_K tr[\int_V \int_K f(w,z)\overline{f(u,y)}e^{-\ i\langle\ \xi,\ y^{-1}(w-u)\rangle}\ \gamma(z^{-1}y)] dwdzdudyd\xi \\
= &\sum_{\gamma\in\widehat{K}} d_\gamma \int_V \int_V \int_K tr[\int_V \int_K f(w,z)\ \overline{f(u,y)}e^{-\ i\langle y\ \xi,\ (w-u)\rangle}\ \gamma(z^{-1})\gamma(y)] dwdzd\xi
\end{aligned}$$

Using the fact that the lebesgue $d\xi$ is invariant by the group rotation $SO(n)$, we get

$$
\begin{aligned}
& \overset{\vee}{f} * f(0,1) \\
= & \sum_{\gamma\in\widehat{K}} d_\gamma \int_V \int_V \int_K tr[\int_V \int_K f(w,z)\ \overline{f(u,y)} e^{-\ i\langle y\ \xi,\ (w-u)\rangle}\ \gamma(z^{-1})\gamma(y)] dwdzd\xi \\
= & \sum_{\gamma\in\widehat{K}} d_\gamma \int_V \int_V \int_K tr[\int_V \int_K f(w,z)\ \overline{f(u,y)} e^{-\ i\langle \xi,\ (w-u)\rangle}\ \gamma(z^{-1})\gamma(y^{-1})^{-1}] dwdzdudyd\xi \\
= & \sum_{\gamma\in\widehat{K}} d_\gamma \int_V \int_V \int_K tr[\int_V \int_K f(w,z) e^{-\ i\langle \xi,\ w\rangle}\ \overline{f(u,y)} e^{-\ i\langle \xi,\ -u\rangle}\ \gamma(z^{-1})\gamma^*(y^{-1})] dwdzdudyd\xi \\
= & \sum_{\gamma\in\widehat{K}} d_\gamma \int_V \int_V \int_K \int_V \int_K tr[f(w,z) e^{-\ i\langle \xi,\ w\rangle}\gamma(z^{-1})\ \overline{f(u,y) e^{-\ i\langle \xi,\ u\rangle}}\ \gamma^*(y^{-1})] dwdzdudyd\xi \\
= & \sum_{\gamma\in\widehat{K}} d_\gamma \int_V tr[T\mathcal{F}f(\xi,\gamma)\ \overline{T\mathcal{F}f(\xi,\gamma^*)}] d\xi = \sum_{\gamma\in\widehat{K}} d_\gamma \int_V \|T\mathcal{F}f(\xi,\gamma)\|^2_{H.S}\, d\xi
\end{aligned}
$$



Hence our theorem
Let $L = V \times K \times K$ be the group with law:

$$(v,x,y)(w,s,t) = (v + \rho(y)w, xs, yt) \tag{21}$$

Let $\mathcal{D}(V \times K \times K\ )$, $\mathcal{S}(V \times K \times K)$, and $C^\infty(V \times K \times K)$ be $C^\infty$ with compact support, the Schwartz space and the space of $C^\infty$- functions of the group $L$.

**Definition 2.2.**. *For any $f \in \mathcal{S}(G)$, we can define a function $\widetilde{f} \in \mathcal{S}(V \times K \times K)$ as follows*

$$\widetilde{f}(v,x,y) = f(\rho(xv), xy) = f(xv.xy) \tag{22}$$

*for all $\xi \in V \simeq V^*$and for all $\gamma \in \widehat{K}$,where $xv$ signifies $\rho(xv)$.* Note here that the function $\widetilde{f}$ is invariant in the following sense

$$\widetilde{f}(tv, xt^{-1}, ty) = \widetilde{f}(v,x,y) \tag{23}$$

We denote by $\mathcal{D}_K(V \times K \times K\ )$, $\mathcal{S}_K(V \times K \times K)$, and $C_K^\infty(V \times K \times K)$ the spaces $C^\infty$ with compact, the Schwartz space and the space of $C^\infty$- functions of the group $L$, which are invariant in sense (21)

**Definition 2.3.**. *for any two function $f \in \mathcal{S}(G)$ and $F \in \mathcal{S}_K(V \times K \times K)$, we can define a convolution product of $f$ and $F$ on $G$ as*

$$\begin{aligned} f * F(v,x,y) &= \int\limits_G F((w,z)^{-1}(v,x,y)f(w,z)dwdz \\ &= \int\limits_G F(z^{-1}(v-w),x,z^{-1}y)f(w,z)dwdz \end{aligned} \tag{24}$$

This leads to obtain

**Lemma 2.1**. *If $F$ is invarant in sense (21), then we get*

$$f * F(v,x,y) = f *_c F(v,x,y) \tag{25}$$

*for every $f \in C^\infty(V \times K)$, $(v,x,y) \in L$, where $*$ signifies the convolution product on $G = V \times \{I_K\} \times K$ with respect the variables $(v,y)$ and $*_c$signifies the convolution product on the subgroup $B = V \times K \times \{I_K\}$ of $L$,which is the direct product of $V$ and $K$ with respect the variables $(v,x)$*

*Proof:* Let $f \in \mathcal{S}(G)$ and $F \in \mathcal{S}_K(V \times K \times K)$, then we have

$$\begin{aligned} f * F(v,x,y) &= \int\limits_G F((w,z)^{-1}(v,x,y)f(w,z)dwdz \\ &= \int\limits_G F(z^{-1}(v-w),x,z^{-1}y)f(w,z)dwdz \\ &= \int\limits_G F((v-w),xz^{-1},y)f(w,z)dwdz \\ &= F *_c f(v,x,y) \end{aligned} \tag{26}$$

So the lemma is proved.

**Definition 2.4**. *If $f \in \mathcal{S}(G)$, one can define the Fourier transform of its invariant $\widetilde{f}$ as*

$$T\mathcal{F}\widetilde{f}(\xi,\gamma,1) = \int\limits_V \int_K \sum_{\delta \in \widehat{K}} d_\delta tr[\int_K T\mathcal{F}\widetilde{f}(v,x,y)\delta(y^{-1})dy]\gamma(x^{-1})dxe^{-i\langle \xi, v\rangle}dv \tag{27}$$

where $\widetilde{f}$ is the function defined by

$$\widetilde{f}(v,x,y) = f(xv,xy) \tag{28}$$

**Theorem 2.2**. *For any two functions $g$ and $f$ belong to $C^\infty(V \times K) = C^\infty(G)$ , then we have*

$$T\mathcal{F}(g * \widetilde{f})(\xi,\gamma,1) = T\mathcal{F}(\widetilde{f} *_c g)(\xi,\gamma,1) = \mathcal{F}(\widetilde{f})(\xi,\gamma,1)T\mathcal{F}(g)(\xi,\gamma) \tag{29}$$

*Proof:* By lemma 2.1. we have if $f$ and $g$ two functions from $\mathcal{S}(G)$

$$\begin{aligned} & T\mathcal{F}(g * \widetilde{f})(\xi,\gamma,1) \\ = & \int_V \int_K \sum_{\delta \in \widehat{K}} d_\delta tr[(g * \widetilde{f})(v,x,\delta)]\gamma(x^{-1})dxe^{-i\langle \xi, v\rangle}dv \\ = & \int_V \int_K \sum_{\delta \in \widehat{K}} d_\delta tr[\int_K ((g * \widetilde{f})(v,x,y))\delta(y^{-1})dy]\gamma(x^{-1})dxe^{-i\langle \xi, v\rangle}dv \\ = & \int_V \int_K \sum_{\delta \in \widehat{K}} d_\delta tr[\int_K ((\widetilde{f} *_c g)(v,x,y))\delta(y^{-1})dy]\gamma(x^{-1})dxe^{-i\langle \xi, v\rangle}dv \end{aligned} \tag{30}$$

Changing variables $v - u = w, xt^{-1} = z$, this implies

$$\begin{aligned} & T\mathcal{F}(g * \widetilde{f})(\xi,\gamma,1) \\ = & \int_V \int_K \int_V \int_K \widetilde{f}(v-u, xt^{-1},1)]g(u,t)dudt\gamma(x^{-1})dxe^{-i\langle \xi, v\rangle}dv \\ = & \int_V \int_K \int_V \int_K \widetilde{f}(w,z,1)]g(u,t)\gamma(t^{-1}z^{-1})dxe^{-i\langle \xi, w+u\rangle}dudtdwdzvdx \\ = & \int_V \int_K \int_V \int_K \widetilde{f}(w,z,1)]g(u,t)\gamma(z^{-1})\gamma(t^{-1})dzdte^{-i\langle \xi, w\rangle}e^{-i\langle \xi, u\rangle}dudw \\ = & \int_V \int_K \int_V \int_K \widetilde{f}(zw,z)]g(u,t)\gamma(z^{-1})\gamma(t^{-1})dzdte^{-i\langle \xi, w\rangle}e^{-i\langle \xi, u\rangle}dudw \end{aligned} \tag{31}$$

$$= T\mathcal{F}\widetilde{f}(\xi,\gamma,1)T\mathcal{F}g(\xi,\gamma) \tag{32}$$

**Theorem 2.3.** (***Plancheral's formula***) *For any* $f \in L^1(G) \cap L^2(G)$,*we get*

$$f \ast \overset{\widetilde{\vee}}{f}(0,1) = \int\limits_G |f(v,x)|^2 \, dvdx = \sum_{\gamma \in \widehat{K}} d_\gamma \int\limits_V \|T\mathcal{F}f(\xi,\gamma)\|_2^2 \, d\xi \qquad (33)$$

*where*

$$\overset{\widetilde{\vee}}{f}(v,x,y) = \overline{f((xv,xy)^{-1})} \qquad (34)$$

The proof of this theorem results immediatly from 2.2

# 3 Fourier Transform and Differential Operators on the Motion Groups

**3.** We denote by $L^1(G)$ the Banach algebra that consists of all complex valued functions on the group $G$, which are integrable with respect to the Haar measure of $G$ and multiplication is defined by convolution on $G$ , and we denote by $L^2(G)$ the Hilbert space of $G$. So we have for any $f \in L^1(G)$ and $\phi \in L^1(G)$

$$\phi \ast f(v,x) = \int\limits_G f((w,y)^{-1}(v,x))\phi(w,y)dwdy \qquad (35)$$

where $dwdy$ is the Haar measure on $G$ , and $\ast$ denotes the convolution product on $G$. Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra of $G$; which is canonically isomorphic onto the algebra of all distributions on $G$ supported by $\{I_G\}$, where $\{I_G\}$ is the identity element of $G$. For any $u \in \mathcal{U}$ one can define an left invariant differential operator $P$ on $G$ as follows:

$$P_u f(X) = u \ast f(v,x) = \int\limits_G f((w,y)^{-1}(v,x))u(w,y)dwdy \qquad (36)$$

for any $f \in C^\infty(G)$. The mapping $u \to P_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on $G$.

where $\ast_c$ means the convolution product on the group $A$, and $dwdy$ is the Haare measure on $A$. We denote by $\mathcal{U}_B$ the complexified universal enveloping algebra of the real Lie algebra of $B$,which is canonically isomorphic onto the algebra of all distributions on $B$ supported by the $\{I_B\}$, where $\{I_B\}$ is the identity element of $A$. For any $u \in \mathcal{U}$ one can define a differential operator with constant coefficients $Q$ on $B$ as

$$f *_c Q_u(v,x) = f *_c u(v,x) = \int\limits_B f(v-w, xy^{-1})u(w,y)dwdy \qquad (37)$$

for any $f \in C^\infty(A)$, where $*_c$ means the convolution product on the group $B$.The mapping $u \to Q_u$ is an algebra isomorphism of $\mathcal{U}_B$ onto the algebra of all invariant differential operators on $B$. For more details see [4, 11].

**Theorem 3.1.** *Let $P$ be a right invariant differential on the motion group $G$, then $P$ has a fundamental solution if and only if there is a constant $C$ and a number $q \in N$, such that*

$$\det\ T\mathcal{F}u(0,\gamma) \neq 0,\ and \ \left\| \frac{{}^{co}T\mathcal{F}u(0,\widetilde{\gamma)}}{\det\ T\mathcal{F}u(0,\widetilde{\gamma)}} \right\| \leq d_\gamma(q) \qquad (38)$$

*Proof:* Let $u$ be the distribution associted to the right invariant differential operator $P$. Let $End(E_\gamma)$ be the space of all enomorphisms of $E_\gamma$ and let $P_\zeta$ be the polynomial valued in $End(E_\gamma)$, defined by

$$(\zeta,\gamma) \mapsto T\mathcal{F}(\overset{\vee}{u})((\xi,\gamma)+(\zeta,\gamma)) = T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma) \qquad (39)$$

For any $f \in \mathcal{D}(G)$, we put

$$\langle S, f\rangle = \int\limits_{\mathbb{V}} \int\limits_{\Omega} \sum_{\gamma\in\widehat{K}} d_\gamma tr[\frac{{}^{co}T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)}{\det\ T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)} T\mathcal{F}(f)(\xi+\zeta,\gamma)]\Phi(P_\zeta,\xi)d\zeta d\pi d\omega d\xi d\lambda(n) \qquad (40)$$

where $\Omega$ is a ball in $\mathbb{C}^n$ with center 0, $\Phi$ is Hormander,s function[12]. Then by [2, 573−579 ], $S$ defines distribution on $G$, so we can define a new distribution $\widetilde{S}$ assiciated to $S$ as

$$
\begin{aligned}
& \langle \widetilde{S},\ f\ \rangle \\
= & \ \langle S,\ \widetilde{f}\ \rangle \\
= & \int_{\mathbb{V}} \int_{\Omega} \sum_{\gamma \in \widehat{K}} d_\gamma tr[\frac{{}^{co}T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)}{\det\ T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)} T\mathcal{F}(\widetilde{f})(\xi+\zeta,\gamma,I)]\Phi(P_\zeta,\xi)d\zeta d\xi
\end{aligned}
$$

where $\Phi(P_\zeta,\xi)$ is the Hormander function [12]. Then we have by [2, 573 − 579 ] and by Hormander construction , we have

$$
\begin{aligned}
& \langle \widetilde{u*S},\ f\ \rangle \\
= & \ \langle u*S,\ \widetilde{f}\ \rangle = \langle S, \overset{\vee}{u} *\ \widetilde{f}\ \rangle \\
= & \int_{\mathbb{V}} \int_{\Omega} \sum_{\gamma \in \widehat{SO(3)}} d_\gamma tr[\frac{{}^{co}T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)}{\det\ T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)} T\mathcal{F}(\overset{\vee}{u}*\widetilde{f})(\xi+\zeta,\gamma,I))]\Phi(P_\zeta,\xi)d\zeta d\xi \\
= & \int_{\mathbb{V}} \int_{\Omega} \sum_{\gamma \in \widehat{K}} d_\gamma tr[\frac{{}^{co}T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)}{\det\ T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)} T\mathcal{F}(\overset{\vee}{u}*_c\widetilde{f})(\xi+\zeta,\gamma,I)]\Phi(P_\zeta,\xi)d\zeta d\xi \\
= & \int_{\mathbb{V}} \int_{\Omega} \sum_{\gamma \in \widehat{K}} d_\gamma tr[\frac{{}^{co}T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)}{\det\ T\mathcal{F}(\overset{\vee}{u})(\xi+\zeta,\gamma)} T\mathcal{F}(\widetilde{f})(\xi+\zeta,\gamma,I)]\Phi(P_\zeta,\xi)d\zeta d\xi \\
= & \int_{\mathbb{V}} \int_{\Omega} \sum_{\gamma \in \widehat{K}} d_\gamma tr[T\mathcal{F}(\widetilde{f})(\xi+\zeta,\gamma,I)]\Phi(P_\zeta,\xi)d\zeta d\xi \\
= & \int_{\mathbb{V}} \sum_{\gamma \in \widehat{K}} d_\gamma tr[T\mathcal{F}(\widetilde{f})(\xi,\gamma,I)]d\xi = \widetilde{f}(0,I,I) = \langle \delta_G, \widetilde{f} \rangle \qquad (41)
\end{aligned}
$$

Consequently, we have

$$
u * S(v,q) = \delta_G(v,q) \qquad (42)
$$

Hence the proof of our theorem.

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